Branes, Quantum Nambu Brackets, and the Hydrogen Atom

Cosmas Zachos

High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439-4815, USA
zachos@hep.anl.gov

Thomas Curtright

Department of Physics, University of Miami, Coral Gables, FL 33124-8046, USA
curtright@physics.miami.edu

Received 8/2004

The Nambu Bracket quantization of the Hydrogen atom is worked out as an illustration of the general method. The dynamics of topological open branes is controlled classically by Nambu Brackets. Such branes then may be quantized through the consistent quantization of the underlying Nambu brackets: properly defined, the Quantum Nambu Brackets comprise an associative structure, although the naive derivation property is mooted through operator entwinement. For superintegrable systems, such as the Hydrogen atom, the results coincide with those furnished by Hamiltonian quantization—but the method is not limited to Hamiltonian systems.

PACS: 02.30.Ik, 11.30.Rd
Key words: Nambu Brackets, quantization, integrability

This talk by the first author is well-covered by the writeup in ref [1]. Here, instead, by way of appendiceal illustration, we briefly extend Pauli's celebrated quantization of the Hydrogen atom [2] to quantization by Quantum Nambu Brackets (QNB) detailed in ref [3], in straightforward application of results in that work.

The classical motion of topological open membranes as well as maximally superintegrable systems [1] (indeed, most of the maximally symmetric systems solved in introductory physics!) is controlled by Classical Nambu Brackets (CNB), the multilinear generalization of Poisson Brackets (PB) [4]. Maximally superintegrable systems, are, of course, also described by conventional Hamiltonian mechanics classically, and are also quantized in standard fashion.

Consider the all-familiar classical Coulomb problem, or, with a view to its impending quantization, the Hydrogen-atom problem [2]. Hamilton's equations of motion in phase space are

$$\frac{dz^i}{dt} = \{z^i, H\}, \tag{1}$$

with z^i standing for the phase-space 6-vector $(\mathbf{r}, \mathbf{p})$, and

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r}\ , \tag{2}$$

in simplified (rescaled) notation.

The invariants of the hamiltonian are the angular momentum vector,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{3}$$

and the Hermann-Bernoulli-Laplace vector [5], now usually called the Pauli-Runge-Lenz vector,

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}}. \tag{4}$$

(Dotting it by $\hat{\mathbf{r}}$ instantly yields Kepler's elliptical orbits, $\hat{\mathbf{r}} \cdot \mathbf{A} + 1 = \mathbf{L}^2/r$.)

Since $\mathbf{A} \cdot \mathbf{L} = 0$, it follows that

$$H = \frac{\mathbf{A}^2 - 1}{2\mathbf{L}^2} \quad . \tag{5}$$

However, to simplify the PB Lie-algebraic structure,

$$\{L_i, L_j\} = \epsilon^{ijk} L_k, \qquad \{L_i, A_j\} = \epsilon^{ijk} A_k, \qquad \{A_i, A_j\} = -2H\epsilon^{ijk} L_k, \tag{6}$$

it is useful to redefine $\mathbf{D} \equiv \frac{\mathbf{A}}{\sqrt{-2H}}$, and further

$$\mathcal{R} \equiv \mathbf{L} + \mathbf{D}, \qquad \mathcal{L} \equiv \mathbf{L} - \mathbf{D}. \tag{7}$$

These six simplified invariants obey the standard $SU(2)\times SU(2) \sim SO(4)$ symmetry algebra,

$$\{\mathcal{R}_i, \mathcal{R}_j\} = \epsilon^{ijk} \mathcal{R}_k, \qquad \{\mathcal{R}_i, \mathcal{L}_j\} = 0, \qquad \{\mathcal{L}_i, \mathcal{L}_j\} = \epsilon^{ijk} \mathcal{L}_k, \tag{8}$$

and depend on each other and the hamiltonian through

$$H = \frac{-1}{2\mathcal{R}^2} = \frac{-1}{2\mathcal{L}^2} \ , \tag{9}$$

so only five of the invariants are algebraically independent.

Equivalently to the law of motion (1), however, the same classical evolution may also be specified by Nambu's equation of motion [4], (as is the case for all superintegrable systems [6]),

$$\frac{dz^i}{dt} = H^2 \ \{z^i, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\} \ . \tag{10}$$

The object on the right-hand side multiplied by H^2 is a 6-CNB, i.e. a phase-space Jacobian determinant (volume element),

$$\{I_1, I_2, I_3, I_4, I_5, I_6\} \equiv \frac{\partial(I_1, I_2, I_3, I_4, I_5, I_6)}{\partial(x, p_x, y, p_y, z, p_z)} \ . \tag{11}$$

It is Nambu's [4] celebrated completely antisymmetric multilinear generalization of PBs, and, like all even-CNBs, it amounts to the Pfaffian [1] of the (antisymmetric) matrix with elements $\{I_i, I_j\}$,

$$\{I_1, I_2, I_3, I_4, I_5, I_6\} = \frac{\epsilon^{ijklmn}}{48} \{I_i, I_j\}\{I_k, I_l\}\{I_m, I_n\}, \tag{12}$$

i.e., it resolves into a sum of products of PBs, specified uniquely by complete antisymmetry and linearity in all arguments I_i [3].

By utilizing properties of the determinant, such as combining columns, and Leibniz's rule of differentiation, one easily finds several equivalent expressions of (10), as was first worked out in ref [7]. It was suggested in that reference that inclusion of the hamiltonian itself among the invariants in the arguments of the CNB might be problematic, but, in fact, it is quite straightforward: it follows directly from (10) that, alternatively,

$$\frac{dz^i}{dt} = \frac{1}{4\mathcal{R}_3\mathcal{L}_3}\{z^i, H, \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\} \ . \tag{13}$$

This form and the resolution into PBs (12) serves, by (8), to instantly prove equivalence of (10) to (1). Motion is evidently confined on the constant surfaces specified by the five invariants entering in the CNB—a generic feature of the CNB description of maximally superintegrable systems [1, 3, 6, 7, 8]; any algebraically independent invariants would do.

Actually, this problem has already been addressed in the treatment of S^3 in ref [3], eqns (56,61), except that the respective hamiltonian in that problem is the inverse of the present Coulomb one.

The action whose extremization yields this evolution law is a topological 5-form action

$$S = \int \Big(x\ dp_x \wedge dy \wedge dp_y \wedge dz \wedge dp_z + \ln(\mathcal{R}_3 + \mathcal{L}_3)\ d\mathcal{R}_1 \wedge d\mathcal{R}_2 \wedge d\mathcal{L}_1 \wedge d\mathcal{L}_2 \wedge dt \Big) \ , \tag{14}$$

a Cartan integral invariant "4-brane" action analogous to $(4+1)$-dimensional σ-model WZWN topological interaction terms [1, 9]. It originates by Stokes' law in the integral on an open 6-surface of the exact 6-form

$$\begin{aligned} d\omega_5 &= dx \wedge dp_x \wedge dy \wedge dp_y \wedge dz \wedge dp_z + d\ln(\mathcal{R}_3 + \mathcal{L}_3) \wedge d\mathcal{R}_1 \wedge d\mathcal{R}_2 \wedge d\mathcal{L}_1 \wedge d\mathcal{L}_2 \wedge dt \\ &= (dx - \{x, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}dt) \wedge (dp_x - \{p_x, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}dt) \\ &\wedge (dy - \{y, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}dt) \wedge (dp_y - \{p_y, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}dt) \\ &\wedge (dz - \{z, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}dt) \wedge (dp_z - \{p_z, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}dt) \, . \end{aligned} \tag{15}$$

For any function of phase space with no explicit time dependence, then, the classical evolution law

$$\frac{df}{dt} = H^2\{f, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\} \tag{16}$$

is to be quantized ($\hbar$-deformed) consistently below, as detailed for S^3 in refs [1, 3].

As noted by Pauli, extension to operators requires a hermitean version of (4),

$$\mathbf{A}' = \tfrac{1}{2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \hat{\mathbf{r}}, \tag{17}$$

so that

$$(\mathbf{A}')^2 = 2H(\mathbf{L}^2 + \hbar^2) + 1, \tag{18}$$

leading to $\mathbf{D}' \equiv \frac{\mathbf{A}'}{\sqrt{-2H}}$, and further to the respective chiral reduction $\mathcal{R}'$ and $\mathcal{L}'$, which obey

$$[\mathcal{R}'_i, \mathcal{R}'_j] = 2i\hbar\ \epsilon^{ijk}\mathcal{R}'_k, \qquad [\mathcal{R}'_i, \mathcal{L}'_j] = 0, \qquad [\mathcal{L}'_i, \mathcal{L}'_j] = 2i\hbar\ \epsilon^{ijk}\mathcal{L}'_k, \tag{19}$$

and hence

$$H = \frac{-1}{2(\mathcal{R}'^2 + \hbar^2)} = \frac{-1}{2(\mathcal{L}'^2 + \hbar^2)}\ . \tag{20}$$

One may thus omit the primes on the operator expressions without appreciable loss of clarity, and recall the eigenvalues of the quadratic Casimir invariants of $SU(2)$ for $s = 0, \frac{1}{2}, 1, \ldots$, leading to the Balmer spectrum for the hamiltonian,

$$\langle H \rangle = \frac{-1}{2\hbar^2(4s(s+1)+1)} = \frac{-1}{2\hbar^2(2s+1)^2}\ . \tag{21}$$

The size of these $SU(2) \times SU(2)$ multiplets, $(2s+1)^2$, is the corresponding degeneracy.

Time evolution in the hamiltonian picture is given by Heisenberg's quantum equation of motion,

$$i\hbar \frac{df}{dt} = [f, H]\ ; \tag{22}$$

in the QNB picture (detailed in refs [1, 3]) it works as outlined below.

A 6-QNB, $[I_1, I_2, I_3, I_4, I_5, I_6]$, consists of of the fully antisymmetrized linear product of its 6 operator arguments. Analogously to its 6-CNB counterpart, it can be shown to resolve to a sum of strings of commutators,

$$[I_1, I_2, I_3, I_4, I_5, I_6] = \frac{\epsilon^{ijklmn}}{8}[I_i, I_j][I_k, I_l][I_m, I_n]. \tag{23}$$

This is longer than its classical counterpart (which has only $15 = 5 \cdot 3$ distinct terms), as commutators need not commute with each other in general, so their symmetric entwinement leads to $90 = 3!5 \cdot 3$ terms. In practical terms, in general, evaluation of even QNBs resolves to judicious evaluation of commutators [3]. It is evident by inspection of this resolution that the classical limit ($\hbar \to 0$) of this QNB is the CNB discussed,

$$[I_1, I_2, I_3, I_4, I_5, I_6]\ \rightarrow\ 3!(i\hbar)^3\ \{I_1, I_2, I_3, I_4, I_5, I_6\}. \tag{24}$$

It follows then from (23) (cf eqns (183)+(184) of [3]) that, for the particular simple Lie algebras (19), the quantization of (16) is just

$$3(i\hbar)^3\left((\mathcal{R}_3 + \mathcal{L}_3)\frac{df}{dt} + \frac{df}{dt}(\mathcal{R}_3 + \mathcal{L}_3)\right) = H\Big[f, \mathcal{R}_3 + \mathcal{L}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\Big]H + \mathcal{Q}(O(\hbar^5)). \tag{25}$$

$\mathcal{Q}(O(\hbar^5))$ is a subdominant nested commutator "quantum rotation" [3], vanishing in the classical limit[1]). Solving for df/dt may be more challenging technically

[1]) Specifically, $\mathcal{Q} = 2\hbar^2 H \sum_i ([[[f, \mathcal{L}_i], \mathcal{L}_i], \mathcal{R}_3] + [[[f, \mathcal{R}_i], \mathcal{R}_i], \mathcal{L}_3])H$.

(the Jordan-Kurosh spectral problem), but the formulation is still equivalent to the standard Hamiltonian quantization of this problem [3]. Of course, expectation values in sectors with definite $L_3 = (\mathcal{L}_3 + \mathcal{R}_3)/2$ are thus proportional to $\langle \frac{df}{dt} \rangle$.

Similarly, (cf eqn (77) of ref [3], also for S^3), equivalent forms such as (13) quantize through

$$4(i\hbar)^3 \left(\mathcal{R}_3, \mathcal{L}_3, \frac{df}{dt} \right) = \Big[f, H, \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2 \Big], \tag{26}$$

where the parenthesis on the left indicates the complete symmetrization of its three arguments.

In contrast to Heisenberg's law of motion in the Hamiltonian formulation, the operator acting on f in the QNB formulation above is not a derivative operator, i.e., it does not obey Leibniz's chain rule, because the actual (time) derivatives are entwined with other operators. This was all too widely thought to be an obstacle in utilizing QNBs which are not derivative operators themselves, but it was shown to not be a consistency problem at all, for even-QNBs[2] [3, 1]. In quantization, associativity trumps naive derivation features[3].

This general methodology has proven successful in a large number of systems, including some non-Hamiltonian ones [1].

Thanks are due to C Burdik and the organizers of the International Colloquium. This work was supported by the US Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38, and the NSF Award 0303550.

References

[1] T Curtright and C Zachos, 'Branes, strings, and odd quantum Nambu Brackets', in: *Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries (QTS3), 2003*, World Scientific Publishing, Singapore, 2004, [hep-th/0312048].

[2] W Pauli, Z Phys **36** (1926) 336-363;
for an outstanding modern review, see G Valent, Am J Phys **71** (2003) 171-175, [quant-ph/0212010].

[3] T Curtright and C Zachos, Phys Rev **D68** (2003) 085001, [hep-th/0212267]; also see New J Phys **4** (2002) 83.1-83.16, [hep-th/0205063].

[4] Y Nambu, Phys Rev **D7** (1973) 2405-2412.

[5] H Goldstein, Am J Phys **44** (1976) 1123-1124.

[6] C Gonera and Y Nutku, Phys Lett **A285** (2001) 301-306;
Y Nutku, J Phys **A36** (2003) 7559-7567, [quant-ph/0306059].

[2]) Odd-QNBs may be defined consistently through even ones of higher rank.

[3]) As a formal wisecrack, consider defining the following antisymmetric bracket for a fixed operator 0, $[\![A, B]\!] \equiv AOB - BOA$. This clearly satisfies the Jacobi identity, but it fails a naive derivation property, as $[\![A, BC]\!] \neq [\![A, B]\!]C + B[\![A, C]\!]$. Of course, suitable insertions of O in all products would restore that property here.

[7] R Chatterjee, Lett Math Phys **36** (1996) 117-126.

[8] A Teğmen and A Verçin, Int J Mod Phys **A19** (2004) 393-410, [math-ph/0212070].

[9] F Estabrook, Phys Rev **8** (1973) 2740-2743;
F Lund and T Regge, Phys Rev **D14** (1976) 1524-1535;
L Takhtajan, Commun Math Phys **160** (1994) 295-316;
J Hoppe, Helv Phys Acta **70** (1997) 302-317, [hep-th/9602020];
Y Matsuo and Y Shibusa, JHEP02 (2001) 006, [hep-th/0010040];
B Pioline, Phys Rev **D66** (2002) 025010;
A De Castro et al., Phys Lett **B584** (2004) 171-177, [hep-th/0306094];
M Sheikh-Jabbari, [hep-th/0406214].